CONVERGENCE OF POINCARÉ SERIES WITH TWO COMPLEX COWEIGHTS

PAUL C. PASLES

Abstract. We examine convergence properties of Poincaré series with two complex coweights and establish a converse to the well-known result of Petersson regarding absolute convergence of such series.

1. Motivation

In his classic tome on the subject, J. Lehner enumerates three goals in the study of analytic automorphic forms ([5], p.155, referring in part to [10]):

1. Prove the existence of nonconstant automorphic forms of each [weight] and develop their properties.
2. Provide analytical expressions (e.g., Poincaré series) which are automorphic forms.
3. Find a family of such analytical invariants which spans or at least is dense in the linear space of automorphic forms of a given [weight].

For *nonanalytic* forms of arbitrary complex coweights, the first two goals are realized for the theta group in [7]. In the current paper we explain the difficulty inherent in the successful completion of this tripartite plan.

2. Preliminaries

We will be concerned with automorphicity on the *modular* and *theta* groups, each of which is a special instance of a *Hecke group*.

Definition 2.1. *For* $\lambda > 0$, *the* Hecke group $\mathcal{G}_\lambda$ *is defined by*

$$\mathcal{G}_\lambda = \langle S_\lambda, T \rangle \,,$$

the 2×2 *matrix group generated by*

$$S_\lambda = \left[\begin{array}{cc} 1 & \lambda \\ 0 & 1 \end{array} \right] \text{ and } T = \left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right].$$

$\mathcal{G}_\lambda$ acts on the complex upper half-plane $\mathcal{H} = \{z : \text{Im } z > 0\}$ by linear fractional transformations:

$$Mz = \frac{az+b}{cz+d} \text{ for } M = \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] \in \mathcal{G}_\lambda, \; z \in \mathbb{H}.$$

(Note that the distinct matrices $M, -M$ represent the same transformation.) In

Date: August 1, 1998.
1991 *Mathematics Subject Classification.* Primary 11F37; Secondary 11F11, 11F12, 30B50.
Key words and phrases. automorphic forms, Poincaré series.
This paper is in final form and no version of it will be submitted for publication elsewhere.

the current work, we will focus on $\mathcal{G}_1$ and $\mathcal{G}_2$, known as the *modular* and *theta* groups, respectively.

Exponentiation is to be interpreted using certain argument conventions which we now delineate. For $z, w \in \mathbb{C}$, $z \neq 0$, define $z^w = e^{w \log z}$. Here, $\log z = \log |z| + i \arg z$, where $\log |z|$ is the principal branch ($\log 1 = 0$), and

$$-\pi \leq \arg z < \pi.$$

We will make exceptions to this "unitary" convention whenever a binary convention is more convenient (see below).

Definition 2.2. *Let $\alpha, \beta \in \mathbb{C}$, $\lambda > 0$. We call $\upsilon : \mathcal{G}_\lambda \to \mathbb{C} \setminus \{0\}$ a* multiplier system *on $\mathcal{G}_\lambda$ of coweights α, β if $|\upsilon(S_\lambda)| = 1$ and υ satisfies the* consistency condition

$$\begin{aligned}&\upsilon(M_3)(c_3 z + d_3)^\alpha (c_3 \bar{z} + d_3)^\beta \\ &\quad = \upsilon(M_1)(c_1 M_2 z + d_1)^\alpha (c_1 M_2 \bar{z} + d_1)^\beta \upsilon(M_2)(c_2 z + d_2)^\alpha (c_2 \bar{z} + d_2)^\beta,\end{aligned}$$

for all $M_1, M_2 \in \mathcal{G}_\lambda$ such that $M_1 M_2 = M_3$, $M_j = \begin{bmatrix} a_j & b_j \\ c_j & d_j \end{bmatrix}$ for $j = 1, 2, 3$, and all $z \in \mathcal{H}$. Here we observe the binary *argument convention of the Petersson-Maass tradition* [6]:

$$-\pi \leq \arg(cz + d) < \pi$$

$$-\pi < \arg(c\overline{z} + d) \leq \pi,$$

for $z \in \mathcal{H}$, $(c, d) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

That such multiplier systems actually *exist* for any given complex pair α, β has been shown for the theta group [7]; for example, if α and β are arbitrary complex numbers, then one such multiplier system on $\mathcal{G}_2$ of coweights α, β is generated by

$$\upsilon(S_2) = 1, \ \upsilon(T) = e^{\pi i (\beta - \alpha)/2},$$

and the consistency condition.

Considering complex weights or coweights presents a significant complication to automorphic forms theory. In the traditional situation (single real weight, e.g. [1]), a multiplier system has absolute value identically one, and $|\upsilon|$ is a homomorphism on $\mathcal{G}_\lambda$. Unfortunately, as Petersson observed [11], multiplier systems do not have absolute value identically one when the weight (or, in our setting, the difference of the coweights) is nonreal. Unboundedness of υ *seems* to be the underlying cause for divergence of the absolute Poincaré series in the case $\alpha - \beta \notin \mathbb{R}$ (Theorem 3.1); however, as we shall explain in the last section, the reason is still deeper than this.

Another unexpected complication is that, while in the familiar case $|\upsilon|$ is a homomorphism, it is easy to show that this remains true in the current setting if and only if $\alpha - \beta \in \mathbb{R}$. (In one direction this statement is completely obvious, and the other way follows by taking $M_1 = M_2 = T$ in the consistency condition.) Thus, in the general situation with complex coweights which we consider here, life has become less simple.

We begin by calculating several special values of υ which will be needed to prove the main result.

Lemma 2.1. *Let υ be a nonvanishing complex function on $\mathcal{G}_\lambda$ satisfying the consistency condition in coweights α, β. Then:*

(i) $\upsilon(I) = 1$;

(ii) $\upsilon(-I) = \pm e^{i\pi(\alpha-\beta)}$;

(iii) $\upsilon(T) = \pm i^{\beta-\alpha+r}$, *where* $r = \begin{cases} 0 & \text{if } \upsilon(-I) = e^{i\pi(\alpha-\beta)} \\ 1 & \text{if } \upsilon(-I) = -e^{i\pi(\alpha-\beta)}; \end{cases}$

(iv) Therefore, $|\upsilon| \equiv 1 \Leftrightarrow \alpha - \beta \in \mathbb{R}$.

Proof. We will need to observe the binary argument convention carefully.

For *(i)*, apply the consistency condition with $M_1 = M_2 = I$, recalling that υ is nonvanishing.

(ii): Put $M_1 = M_2 = -I$ to get $1 = [\upsilon(-I)]^2 \left(e^{-i\pi\alpha+i\pi\beta}\right)^2$, or $\upsilon(-I) = \pm e^{i\pi(\alpha-\beta)}$.

(iii): Let $M_1 = M_2 = T$. This gives:

$$\begin{aligned} \upsilon(-I)\, e^{i\pi(\beta-\alpha)} &= [\upsilon(T)]^2 (-1/z)^\alpha (-1/\overline{z})^\beta z^\alpha \overline{z}^\beta \\ &= [\upsilon(T)]^2 e^{i\pi(\alpha-\beta)}, \end{aligned}$$

so $[\upsilon(T)]^2 = \upsilon(-I)\, e^{2i\pi(\beta-\alpha)}$. Therefore,

$$\upsilon(T) = \begin{cases} \pm e^{\frac{i\pi}{2}(\beta-\alpha)} & \text{if } \upsilon(-I) = e^{i\pi(\alpha-\beta)} \\ \pm e^{\frac{i\pi}{2}(\beta-\alpha+1)} & \text{if } \upsilon(-I) = -e^{i\pi(\alpha-\beta)}. \end{cases}$$

This proves *(iii)*.

(iv): We have

$$|\upsilon(T)| = \left|\pm e^{\frac{i\pi}{2}(\beta-\alpha+r)}\right| = e^{\frac{-\pi}{2}\operatorname{Im}(\beta-\alpha+r)} = e^{\frac{-\pi}{2}\operatorname{Im}(\beta-\alpha)}.$$

Thus, $|\upsilon| \equiv 1 \Leftrightarrow \alpha - \beta \in \mathbb{R}$, since $|\upsilon(S_\lambda)| = 1$ by assumption and $|\upsilon(T)| = 1 \Leftrightarrow \alpha - \beta \in \mathbb{R}$. For if $\alpha - \beta \notin \mathbb{R}$ then $|\upsilon|$ is not identically one, and if $\alpha - \beta \in \mathbb{R}$ then $|\upsilon| = 1$ on the generators of $\mathcal{G}_\lambda$ (and therefore $|\upsilon| \equiv 1$ since $|\upsilon|$ *is* a homomorphism in this case). ■

Remark 2.1. *It should be duly noted that the multiplier systems with two coweights are actually the* same *as those with a single (complex) weight. For, let $\alpha, \beta \in \mathbb{C}$ and let υ be a nonvanishing complex function on $\mathcal{G}_\lambda$. The following are equivalent:*

(i) υ satisfies the consistency condition for coweights α, β;
(ii) υ satisfies the consistency condition for coweights $\alpha+\omega+2k, \beta+\omega+2\ell$ for all $k, \ell \in \mathbb{Z}$, $\omega \in \mathbb{C}$;
(iii) υ satisfies the consistency condition for coweights $\alpha - \beta, 0$.

The proof is a simple calculation using the binary argument convention.

3. Main result

The main result can be stated after two more definitions.

Definition 3.1. *Let* $C, \alpha, \beta, \omega_1, \omega_2, \ldots, \omega_n \in \mathbb{C}$; $\gamma, \lambda > 0$; *and*

$$\{a_{n_1,n_2,m} \mid 0 \leq n_1, n_2 < \infty; 1 \leq m \leq M\} \subseteq \mathbb{C},$$

with

$$\sum_{n_1+n_2=n} |a_{n_1,n_2,m}| = \mathcal{O}\left(n^\gamma\right) \text{ as } n \to +\infty.$$

For $z \in \mathcal{H},\ z = x + iy$, *define*

$$f\left(z\right) = \sum_{m=1}^{M} y^{\omega_m} \sum_{n_1,n_2=0}^{\infty} a_{n_1,n_2,m} e^{2\pi i \lambda^{-1}(n_1 z - n_2 \overline{z})}.$$

Let υ *be a multiplier system on* $\mathcal{G}_\lambda$ *of coweights* α, β *satisfying* $\upsilon\left(S_\lambda\right) = 1$, $\upsilon\left(T\right) = C$. *If*

$$z^{-\alpha}\overline{z}^{-\beta} f\left(-1/z\right) = Cf\left(z\right)$$

for all $z \in \mathcal{H}$, *we call* f *a* nonanalytic automorphic form *of coweights* α, β *and multiplier system* υ *on* $\mathcal{G}_\lambda$. *When* $\lambda = 1$ *or* 2, f *is called a nonanalytic* modular *form.*

The nonanalytic automorphic form (and its stepsibling the nonanalytic automorphic integral) is studied extensively in [7] and [8]. For certain complex α, β Knopp has considered integrals in the case where $\omega_m \in \mathbb{Z}$ and $a_{n_1,n_2,m} = 0$ for $|n_1| + |n_2| > 0$ [3], and he describes a Hecke theorem in one direction relating each integral to a linear combination of Dirichlet series with exponential and gamma factors. For an account of results on forms in the "classical" case — f analytic, $\alpha \in \mathbb{R}$ and $\beta = 0$, and the underlying group is $\mathcal{G}_1$ or a subgroup — see [2]; in this situation α is simply called the *weight* of f, cf. Remark 2.1. (Forms of integral or half-integral weight are of particular interest for their number-theoretic applications.) In a similar vein, Maass studied a type of real-analytic form on horocyclic groups, with $\alpha - \beta$ real and allowing multiplier systems, in connection with eigenvalues of the hyperbolic Laplacian [6].

Definition 3.2. *Let* α, β *be complex numbers with* $\mathrm{Re}\left(\alpha + \beta\right) > 2$, *and let* $\rho \in \mathbb{Z}$, $\lambda \geq 2$. *Suppose that* υ *is a multiplier system on* $\mathcal{G}_\lambda$ *of coweights* α, β *in the sense of Definition 2.2, with* $\upsilon\left(S_\lambda\right) = e^{2\pi i\kappa}$, $0 \leq \kappa < 1$. *For* $z \in \mathcal{H}$, *the* ρ^{th} *nonanalytic parabolic Poincaré series of coweights* α, β *and multiplier system* υ *on* $\mathcal{G}_\lambda$ *is given by the expression*

$$G^{\upsilon,\rho}_{\lambda,\alpha,\beta}\left(z\right) = \sum e^{2\pi i(\rho+\kappa)Mz/\lambda} \upsilon\left(M\right)^{-1} \left(cz + d\right)^{-\alpha} \left(c\overline{z} + d\right)^{-\beta},$$

where this sum is taken over all lower rows in the group ($M = M_{c,d}$ *has lower row* c, d*); this is the same as summing over the quotient group* ${}_{\langle S_\lambda\rangle}\diagdown^{\mathcal{G}_\lambda}$.

If $\alpha - \beta \in \mathbb{R}$, we can put $k = \alpha - \beta$ and $s = 2\beta$ in the preceding definition to obtain the so-called "Poincaré series with Hecke convergence factor," usually viewed as a function in the complex variables z and s — automorphic in z (and real-analytic in $z, \overline{z}$) for certain fixed s, and analytic in s for fixed z, so long as $z \in \mathcal{H}$ and $\mathrm{Re}\ s > 2 - k$; it may be used to compute the Fourier coefficients of modular forms of small weight [12].

Obviously the first step in proving the existence of a basis for the space of entire forms using Petersson's parabolic Poincaré series is to show that said series has the correct transformation law. This is the consequence of a simple fact: when $\alpha-\beta\in\mathbb{R}$, $G^{\upsilon,\rho}_{\lambda,\alpha,\beta}(z)$ is absolutely uniformly convergent on sets of the form

$$\mathcal{H}_\varepsilon=\left\{z\in\mathbb{C}:\mathrm{Im}\ z>\varepsilon,\ |\mathrm{Re}\ z|<\varepsilon^{-1}\right\},\ \varepsilon>0.$$

(That is to say, the *absolute* series is uniformly convergent on $\mathcal{H}_\varepsilon$.) This was proved by Petersson for the case $\beta=0$ in [10] (see also [5]); we merely observe here that the same proof works for $\beta\neq 0$, so long as $\alpha-\beta$ is real. From this we may conclude that $G^{\upsilon,\rho}_{\lambda,\alpha,\beta}$ is a real analytic function of z and $\overline{z}$ (analytic in z, when $\beta=0$) and that it satisfies the transformation law of an automorphic form. We will show that not only does the converse statement hold — that is, absolute uniform convergence fails when $\alpha-\beta\notin\mathbb{R}$ — but in fact the sum in the above expression for $G^{\upsilon,\rho}_{\lambda,\alpha,\beta}(z)$ is not even *pointwise* absolutely convergent when $\alpha-\beta$ is nonreal. This is the essence of the next theorem, which we state for the special cases of the modular and theta groups.

Theorem 3.1. *Let Γ be a group of linear fractional transformations containing $S_2=\begin{bmatrix}1&2\\0&1\end{bmatrix}$ and $T=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$, and let $\alpha,\beta\in\mathbb{C}$ with $\mathrm{Re}\,(\alpha+\beta)>2$ and $\alpha-\beta\notin\mathbb{R}$. If $\upsilon:\Gamma\to\mathbb{C}\backslash\{0\}$ satisfies the consistency condition for coweights α,β with $|\upsilon(S_2)|=1$, then*

$$\sum_{M\in\ \mathrm{Stab}_\Gamma(\infty)\diagdown\Gamma}\left|e^{2\pi i(\rho+\kappa)Mz/\lambda}\upsilon(M)^{-1}(cz+d)^{-\alpha}(c\overline{z}+d)^{-\beta}\right|$$

diverges for all $\rho\in\mathbb{Z}$, $\kappa\in[0,1)$ and $z\in\mathcal{H}$. In particular, the nonanalytic Poincaré series on $\mathcal{G}_1$ and $\mathcal{G}_2$ are not absolutely convergent for $\alpha-\beta\notin\mathbb{R}$. (I.e., Petersson's hypothesis that the weight be real is not merely sufficient for absolute uniform convergence, but also necessary.)

Proof. Put $\upsilon(S_2)=e^{2\pi i\sigma}$. For integer n, an easy induction shows that

$$(S_2T)^n=\begin{bmatrix}1+n&-n\\n&1-n\end{bmatrix}.$$

Thus, the lower rows of $(S_2T)^n$ are distinct, and so

$$\sum_{M\in\ \mathrm{Stab}_\Gamma(\infty)\diagdown\Gamma}>\sum_{M=(S_2T)^n\in\Gamma}.$$

Now, although $|\upsilon|$ is *not* a homomorphism on $\mathcal{G}_\lambda$ (since $\alpha-\beta\notin\mathbb{R}$ — see remarks in Section 2), it is still true that $|\upsilon((S_2T)^n)|=|\upsilon(S_2T)|^n$. For, by the consistency condition and Lemma 2.1,

$$\upsilon(S_2T)=e^{\pi i(2\sigma+\alpha-\beta+r)},\ r\in\{0,1\},$$

and thus,

$$\begin{aligned}&\upsilon\left((S_2T)^{n+1}\right)\\&\quad=\upsilon\left((S_2T)^n(S_2T)\right)\\&\quad=\upsilon\left((S_2T)^n\right)\upsilon(S_2T)\,\frac{\left[n\left(\frac{2z-1}{z}\right)+1-n\right]^\alpha\left[n\left(\frac{2\overline{z}-1}{\overline{z}}\right)+1-n\right]^\beta z^\alpha\overline{z}^\beta}{\left[(n+1)z-n\right]^\alpha\left[(n+1)\overline{z}-n\right]^\beta}\end{aligned}$$

$$= \upsilon\left((S_2T)^n\right) e^{\pi i(2\sigma+\alpha-\beta+r)} \frac{\left[\frac{(n+1)z-n}{z}\right]^\alpha \left[\frac{(n+1)\overline{z}-n}{\overline{z}}\right]^\beta z^\alpha \overline{z}^\beta}{\left[(n+1)z-n\right]^\alpha \left[(n+1)\overline{z}-n\right]^\beta}.$$

By the binary argument convention, then,

$$\upsilon\left((S_2T)^{n+1}\right) = \upsilon\left((S_2T)^n\right) e^{\pi i(2\sigma+\alpha-\beta+r)}\theta,$$

where

$$\begin{aligned} \theta &= \exp\left\{(\alpha-\beta)\left[\log\left(\frac{(n+1)z-n}{z}\right) + \log z - \log\left((n+1)z-n\right)\right]\right\} \\ &= \exp\left\{(\alpha-\beta)L\right\}. \end{aligned}$$

Since Im $z>0$, we also have

$$\mathrm{Im}\left[\frac{(n+1)z-n}{z}\right],\ \mathrm{Im}\ \left[(n+1)z-n\right]>0.$$

Thus,

$$\arg\frac{(n+1)z-n}{z},\ \arg z,\ \arg\left[(n+1)z-n\right] \in (0,\pi).$$

Since

$$\frac{(n+1)z-n}{z}\cdot z = (n+1)z-n,$$

then,

$$\begin{aligned} L &= \log\frac{(n+1)z-n}{z} + \log z - \log\left((n+1)z-n\right) \\ &= i\left[\arg\frac{(n+1)z-n}{z} + \arg z - \arg\left((n+1)z-n\right)\right] \\ &\in \{0, \pm 2\pi i\}; \end{aligned}$$

and since all three arguments are in the range $(0,\pi)$, we have $L=0$. Therefore $\theta = 1$, and so

$$\upsilon\left((S_2T)^{n+1}\right) = \upsilon\left((S_2T)^n\right) e^{n\pi i(2\sigma+\alpha-\beta+r)} \text{ for } n\in\mathbb{Z}^+.$$

Thus,

$$\upsilon\left((S_2T)^n\right) = e^{n\pi i(2\sigma+\alpha-\beta+r)} = \upsilon\left(S_2T\right)^n \text{ for } n\in\mathbb{Z}^+.$$

Also, by the consistency condition,

$$\begin{aligned} &\upsilon(I) = \\ &\quad \upsilon\left((S_2T)^n\right)\upsilon\left((S_2T)^{-n}\right)\cdot \\ &\quad \cdot\left[n\left(S_2T\right)^{-n}z+1-n\right]^\alpha \left[n\left(S_2T\right)^{-n}\overline{z}+1-n\right]^\beta (-nz+1+n)^\alpha (-n\overline{z}+1+n)^\beta \\ &\quad = \upsilon\left((S_2T)^n\right)\upsilon\left((S_2T)^{-n}\right)\cdot \\ &\quad \cdot\left[n\tfrac{(1-n)z+n}{-nz+1+n}+1-n\right]^\alpha \left[n\tfrac{(1-n)\overline{z}+n}{-n\overline{z}+1+n}+1-n\right]^\beta (-nz+1+n)^\alpha (-n\overline{z}+1+n)^\beta \\ &\quad = \upsilon\left((S_2T)^n\right)\upsilon\left((S_2T)^{-n}\right). \end{aligned}$$

Therefore $\upsilon\left((S_2T)^{-n}\right) = \upsilon\left((S_2T)^n\right)^{-1} = \upsilon\left(S_2T\right)^{-n}$ for $n \in \mathbb{Z}^+$, and so

$$\upsilon\left((S_2T)^n\right) = \upsilon\left(S_2T\right)^n \text{ for } n \in \mathbb{Z}.$$

We have, then,

$$\sum_{M \in \ Stab_\Gamma(\infty) \diagdown \Gamma} \left| e^{2\pi i(\rho+\kappa)(Mz)/\lambda} \left[\upsilon\left(M\right)(cz+d)^\alpha \left(c\overline{z}+d\right)^\beta\right]^{-1} \right|$$
$$\geq \sum_{n\in\mathbb{Z}} \left| e^{2\pi i(\rho+\kappa)[(S_2T)^n z]/\lambda} \left[\upsilon\left((S_2T)^n\right)(nz+1-n)^\alpha \left(n\overline{z}+1-n\right)^\beta\right]^{-1} \right|$$
$$= \sum_{n\in\mathbb{Z}} e^{\frac{-2\pi}{\lambda}\mathrm{Im}\left[\frac{(1+n)z-n}{nz+1-n}\right]} \left|\upsilon\left(S_2T\right)\right|^{-n} \left|(nz+1-n)^{-\alpha}\left(n\overline{z}+1-n\right)^{-\beta}\right|$$
$$= \sum_{n\in\mathbb{Z}} e^{\frac{-2\pi}{\lambda}\mathrm{Im}\left[\frac{\left(\frac{1}{n}+1\right)z-1}{z+\frac{1}{n}-1}\right]} \left[e^{\frac{-\pi}{2}\mathrm{Im}(\beta-\alpha)n} |n|^{\mathrm{Re}(\alpha+\beta)} \left|\left(z+\tfrac{1}{n}-1\right)^\alpha \left(\overline{z}+\tfrac{1}{n}-1\right)^\beta\right|\right]^{-1}.$$

The individual terms of this last sum are asymptotic to

$$e^{\frac{n\pi}{2}\mathrm{Im}(\beta-\alpha)} |n|^{-\mathrm{Re}(\alpha+\beta)} \left|(z-1)^\alpha \left(\overline{z}-1\right)^\beta\right|^{-1} ,$$

so (since z is fixed) convergence of this last series is equivalent to convergence of

$$\sum_{n\in\mathbb{Z}} \left[e^{\frac{\pi}{2}\mathrm{Im}(\beta-\alpha)}\right]^n |n|^{-\mathrm{Re}(\alpha+\beta)} .$$

Since $\mathrm{Re}\left(\alpha+\beta\right) > 2$ and $\mathrm{Im}\left(\alpha-\beta\right) \neq 0$, then, we have proved divergence of the original series. (Here we see where the assumption $\mathrm{Im}\left(\alpha-\beta\right) \neq 0$ comes into play.)

To get the last statement of the theorem, take $\lambda = 1$, $\kappa = 0$ and $\Gamma = \mathcal{G}_1$, or else $\lambda = 2$, $\kappa = 0$ and $\Gamma = \mathcal{G}_2$. ■

This shows that the nonanalytic Poincaré series on $\mathcal{G}_1$ or $\mathcal{G}_2$ are not absolutely convergent if $\alpha-\beta \notin \mathbb{R}$. The converse statement (that absolute convergence follows from $\alpha-\beta \in \mathbb{R}$) can be shown easily by adapting the proof for the case $\beta = 0$ given in [5] .

Remark 3.1. *If one allows more general multiplier systems, say without assuming* $|\upsilon\left(S_\lambda\right)| = 1$, *then the condition necessary for this divergence proof to work is now the more complicated requirement*

$$\mathrm{Im}\left(\alpha-\beta\right) + 2\pi^{-1}\log\left|\upsilon\left(S_2\right)\right| \neq 0.$$

However, such multiplier systems do not occur often in the literature and so we do not consider them here.

Thus, Theorem 3.1 shows that the parabolic Poincaré series construction does not work for arbitrary complex coweights α, β. Nevertheless, when the coweight-difference *is* real, this series can be generalized in many useful ways, including one which we believe to be new and which will be explored further in the sequel [9]:

Let $\lambda \geq 2$, $\rho_j \in \mathbb{Z}$ and $\gamma_j \in \mathbb{C}$ for $1 \leq j \leq 4$. Let α, β be complex numbers satisfying the conditions $\mathrm{Re}\left(\alpha+\beta\right) > 2$ and $\alpha - \beta = \sum_{j=1}^{4}\gamma_j \in \mathbb{R}$. Suppose that $\upsilon_j : \mathcal{G}_\lambda \to \mathbb{C}\backslash\left\{0\right\}$ satisfies the consistency condition in coweights $\gamma_j, 0$ for each j. (Thus, by Remark 2.1, $\upsilon = \prod \upsilon_j$ satisfies the consistency condition in coweights

α, β.) Finally, write $\upsilon_j(S_\lambda) = e^{2\pi i \kappa_j}$, $0 \le \mathrm{Re}\,\kappa_j < 1$, and suppose that $\sum_{j=1}^{4} \kappa_j \in \mathbb{R}$. Supressing all parameters, we put

$$f(z) = \sum \frac{e^{\frac{2\pi i}{\lambda}[(\rho_1+\kappa_1)Mz+(\rho_2+\kappa_2)M\overline{z}+(\rho_3+\kappa_3)M(-z)+(\rho_4+\kappa_4)M(-\overline{z})]}}{\upsilon(M)(cz+d)^\alpha(c\overline{z}+d)^\beta},$$

$z \in \mathcal{H}$, summing as before over $M = M_{c,d} \in {}_{\langle S_\lambda\rangle}\backslash^{\mathcal{G}_\lambda}$, that is, over all lower rows of $\mathcal{G}_\lambda$.

Theorem 3.2.
(i) The series defining f is absolutely uniformly convergent on $\mathcal{H}_\varepsilon$ for $\varepsilon > 0$.
(ii) f is a real analytic function in z and $\overline{z}$ for $\mathrm{Im}\,z > 0$.
(iii) $f(z+\lambda) = \upsilon(S_\lambda) f(z)$.
(iv) f satisfies the transformation law

$$z^{-\alpha}\overline{z}^{-\beta} f(-1/z) = \upsilon(T) f(z).$$

However, even if $\upsilon(S_\lambda) = 1$, in general f still is not *quite* a nonanalytic automorphic form in the sense of Definition 3.1, as it lacks a quasi-Fourier expansion of the appropriate type; see [9].

Proof. First, observe that as a consequence of the consistency condition on υ_j, each of the four expressions

$$\frac{e^{\frac{2\pi i}{\lambda}(\rho_1+\kappa_1)Mz}}{\upsilon_1(M)}, \quad \frac{e^{\frac{2\pi i}{\lambda}(\rho_2+\kappa_2)M\overline{z}}}{\upsilon_2(M)}, \quad \frac{e^{\frac{2\pi i}{\lambda}(\rho_3+\kappa_3)M(-z)}}{\upsilon_3(M)}, \quad \frac{e^{\frac{2\pi i}{\lambda}(\rho_4+\kappa_4)M(-\overline{z})}}{\upsilon_4(M)}$$

depends only on the lower row of M. ($M_1, M_2 \in \mathcal{G}_\lambda$ have the same lower row iff $M_1 = S_\lambda M_2$ for some integer n.) Ergo, the product of these four factors is likewise dependent only on the lower row of the matrix M. This shows that f is well-defined.

The proof of *(i)* is completely analogous to the classical case (details of which may be found in [5]). Incidentally, if κ is real and $\lambda = 1$ or 2, the necessity of taking $\alpha - \beta \in \mathbb{R}$ is clear by the *exact* same calculation which appeared in the proof of Theorem 3.1.

Since

$$\mathcal{H} = \bigcup_{\varepsilon>0} \mathcal{H}_\varepsilon,$$

conclusion *(ii)* follows directly from *(i)* and basic analysis. The rest of the theorem follows from absolute convergence and the consistency condition. ■

We note that Remark 3.1 is valid here as well.

4. Final comments

It is interesting to contrast Theorem 3.1 with recent work of Knopp and Mason [4] on a type of generalized modular form which is applicable to algebraic conformal field theory.

In a result which seems contradictory to Theorem 3.1, they showed that for certain congruence subgroups of $\mathcal{G}_1$ — that is to say, subgroups which contain some *principal* congruence subgroup

$$\Gamma(N) = \{V \equiv \pm I \pmod{N}\}, N \in \mathbb{Z}^+$$

— Poincaré series of large enough real weight *do* converge, despite the fact that they have allowed multiplier systems whose absolute value is unbounded (as we did here).

It seemed in the proof of Theorem 3.1 that convergence or divergence of $G^{\rho,\nu}_{\lambda,\alpha,\beta}$ was determined by the growth of υ. Yet, in [4], unbounded multiplier systems are not an impediment to convergence. These two results appear to be at odds with one another.

Nevertheless, the multiplier systems in [4] are "parabolic":

$$\upsilon\,(P) = 1 \text{ whenever } trace\,(P) = \pm 2.$$

Moreover, it is not the growth of the multiplier system *alone* which determines convergence or divergence, but rather that together with the choice of α, β. The forms in [4] have *real* weight, so the proof of Theorem 3.1 is not in dispute.

The construction of a basis for the space of nonanalytic automorphic forms of arbitrary complex coweights remains an open problem.

References

[1] Hecke, Erich. *Über die Bestimmung Dirichletscher Reihen durch ihre Funktionalgleichung.* Math. Ann. **112** (1936): 664-699.

[2] Knopp, Marvin I. *Modular Functions in Analytic Number Theory,* 2nd edition. New York: Chelsea, 1993.

[3] Knopp, Marvin I. *Nonanalytic Modular Integrals.* Unpublished lectures at Bryn Mawr College, June-July 1994.

[4] Knopp, Marvin and Mason, Geoffrey. *Generalized modular forms,* preprint.

[5] Lehner, Joseph. *Discontinuous Groups and Automorphic Functions (Mathematical Surveys and Monographs No. 8),* Providence: American Mathematical Society, 1964.

[6] Maass, Hans. *Lectures on Modular Functions of One Complex Variable (Lectures on Mathematics No. 29),* Bombay: Tata Institute of Fundamental Research, 1964.

[7] Pasles, Paul C. *Nonanalytic automorphic integrals on the Hecke groups.* Acta Arith., to appear.

[8] Pasles, Paul C. *A Hecke correspondence theorem for nonanalytic automorphic integrals,* in preparation.

[9] Pasles, Paul C. and Pribitkin, Wladimir. *A generalization of the Lipschitz summation formula and some applications,* in preparation.

[10] Petersson, Hans. *Theorie der automorphen Formen beliebiger reeler Dimension und ihre Darstellung durch eine neue Art Poincaréscher Reihen.* Math. Ann. **103** (1930): 369-436.

[11] Petersson, Hans. *Automorphe Formen als metrische Invarianten.* Math. Nachr. **1** (1948): 158-212 and 218-257.

[12] Pribitkin, Wladimir. *The Fourier coefficients of modular forms and Niebur modular integrals having small positive weight, I,* to appear.

Current address: 201 South 18th Street, Apartment 1601, Philadelphia, PA 19103-5928
E-mail address: pasles@e-math.ams.org *URL*: http://www.angelfire.com/id/math