Row Coincidences in Nonsingular Binary Matrices

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Summary

We present best possible bounds for the number of coincidences of ones between two distinct rows of a nonsingular binary matrix of constant row sum. The lower bound is shown best possible by examples. We construct two classes of matrices that show the upper bound is best possible.

Résumé

Nous étudions le nombre de coïncidences des chiffres uns dans chaque paire de lignes différentes d'une matrice binaire et inversible dont toutes les lignes contiennent la même quantité d'unités. Nous présentons des bornes pour ce nombre, et nous démontrons que ces bornes peuvent être atteintes. Que la borne inférieure peut être réalisée est démontré par moyen des exemples. Quant à la borne supérieure, nous présentons la construction de deux familles de matrices pour lesquelles cette borne est atteinte.

1 Introduction

In the theories of coding and of combinatorial designs, one is interested in maximal properties of binary matrices, that is, matrices whose entries are in the set $\{0, 1\}$. A classical example is Fisher's inequality [2, p. 129]. As another example, one has the work of Deza [1] on the maximal number of rows both of certain binary matrices and of matrices that are similar to Latin squares. More recently, for a nonsingular binary $v \times v$ matrix A of constant row sum, Marrero [3] considered the values of the inner product of each of row of A with another v-dimensional binary vector. He showed that those inner-product values are all equal only when the binary vector is made up of either all zeros or else all ones; that is, the coincidences of ones between the binary vector and each of the rows of A are all the same only in the two trivial cases.

Motivated by such results, we study in this paper the number of coincidences of ones between two distinct rows of a nonsingular binary matrix of constant row sum. First we obtain bounds on this number, and then we show that those bounds are best possible. The optimality of the upper bound is proved constructively by exhibiting two classes of matrices that meet our requirements.

2 Definitions and a preliminary bound

The subsets $X_1, \ldots, X_v$ of $\{x_1, \ldots, x_v\}$ are called a (v, k, λ)-*design* if

1. $|X_i| = k$ for each $i := 1, \ldots, v$;
2. $|X_i \cap X_j| = \lambda$ for each pair of distinct $i, j := 1, \ldots, v$; and
3. $0 < \lambda < k < v$.

The existence problem for (v, k, λ)-designs remains unsolved. A (v, k, λ)-design is determined by its *incidence matrix*; this is the binary $v \times v$ matrix $A := [a_{ij}]$ defined by

$$a_{ij} := \begin{cases} 1, & \text{if } x_j \in X_i, \\ 0, & \text{otherwise.} \end{cases}$$

In the same manner one can define incidence matrices for other types of designs.

If A is a nonsingular binary matrix of constant row sum, then C denotes *the number of different coincidences of ones between pairs of distinct rows of* A; that is, C is the number of different values found among all the inner products of pairs of distinct rows of A.

A preliminary upper bound on C can be obtained as follows. Let A be a nonsingular binary $v \times v$ matrix of constant row sum k. There are $v-1$ rows below the first row, $v-2$ rows below the second row, and so on. Therefore, since this inner product is commutative, the maximum number of different values possible for the inner products of pairs of distinct rows of A is equal to $(v-1) + \cdots + 1 = (v-1)v/2$; thus, $C \leq (v-1)v/2$. But as v increases, it turns out that this bound soon becomes too large. A better bound can be obtained in terms of k.

3 Results

For a nonsingular binary $v \times v$ matrix A, we present several results. We begin by giving bounds on C and showing that the lower bound is best possible. Then, for maximal C, we determine the number k of ones that are possible in each row of A. Finally, we construct two classes of matrices which show that the upper bound for C is best possible. One of these constructions produces a matrix with maximal C for each $v \geq 2$ and each possible value of k. The other construction produces, for each $v \geq 2$, a class of matrices having the largest possible value of k and maximal C. By means of an example, we will point out that these constructions are different in the sense that a matrix resulting from one construction need not be obtained by permutations of rows or columns from a matrix produced by the other construction. We conclude by commenting on two plausible conjectures that actually turn out to be false.

Theorem 1. *Suppose A is a nonsingular binary $v \times v$ matrix of constant row sum k. Then $1 \leq C \leq k$, and 1 is the best possible lower bound. Moreover, $C = k$ if and only if the values of the inner products of pairs of distinct rows from A are $0, \ldots, k-1$. In particular, $C = k = 1$ if and only if A is the $v \times v$ identity matrix.*

Proof. Let A be a nonsingular binary $v \times v$ matrix of constant row sum k.

Since each row of A has k ones, the inner product of two distinct rows of A cannot exceed k. If such inner product were equal to k, then those two rows would have to be identical, and consequently A would be singular. Therefore, the possible values for the inner product of two distinct rows from A are $0, \ldots, k-1$. Thus, it follows that $1 \leq C \leq k$, and that $C = k$ if and only if $0, \ldots, k-1$ are the actual values of such inner products.

It is clear that $C = k = 1$ if and only if A is the $v \times v$ identity matrix.

That 1 is the best possible lower bound for C is shown by the existence of any (v, k, λ)-design, as well as by the identity matrix of order v. □

Theorem 2. *Suppose A is a nonsingular binary $v \times v$ matrix of constant row sum k. If $C = k$, then $k \leq \lfloor v/2 \rfloor$.*

Proof. Let A be a nonsingular binary $v \times v$ matrix of constant row sum k. If $C = k$, then, from Theorem 1, there must be in A at least one pair of distinct rows whose inner product is equal to 0.

Let $i \in \{1, \ldots, v\}$, and permute the columns of A as necessary so as to insure that the ith row has all ones in the first k columns and, therefore, all zeros in the remaining $v-k$ columns. Suppose $k > \lfloor v/2 \rfloor$, so that $v - k < k$. This means that, in addition to the ith row, each of the other rows in A must have at least one 1 in the first k columns, and thus the inner product of the ith row with any other row is at least one. Therefore, for each $i := 1, \ldots, v$, the inner product of the ith row with any other row will never be zero, a contradiction. Consequently, $k \leq \lfloor v/2 \rfloor$. □

The following Lemma and its proof are well known. We present this material here because we refer to it a few times in the proofs of the next two theorems.

Lemma. *If A is an $n \times n$ matrix and*

$$A = \begin{bmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a \end{bmatrix},$$

then $\det A = (a-b)^{n-1}\{a + b(n-1)\}$.

Proof. One computes $\det A$ by successively performing on A the following operations:

1. subtract the first column from each of the other columns;
2. add each row but the first row to the first row; and
3. expand along the main diagonal.

The result is $\det A = (a-b)^{n-1}\{a + b(n-1)\}$. □

Theorem 3. *For each $v \geq 2$ and each $k := 1, \ \ldots, \ \lfloor v/2 \rfloor$, there exists a nonsingular binary $v \times v$ matrix which has k ones in each row and for which $C = k$. Thus, for these matrices, the upper bound k obtained in Theorem 1 is best possible for C.*

Proof. Let $v \geq 2$, and let $k \in \{1, \ \ldots, \ \lfloor v/2 \rfloor\}$. Define a binary $v \times v$ matrix A by the following two-step construction.

> *Step 1* The top $v-k+1$ rows are defined as follows. The first row has all ones in the first k columns and zeros elsewhere. The next $v-k$ rows are obtained from the first row by shifting the block of k consecutive ones by one column to the right. Thus, the second row has a zero in the first column, then ones in the next k columns, and then zeros in the remaining $v-k-1$ columns; the third row has zeros in the first two columns, then ones in the next k columns, and then zeros in the remaining $v-k-2$ columns; and so on through the $(v-k+1)$th row, which has zeros in the first $v-k$ columns and then ones in the remaining k columns.
>
> *Step 2* The bottom $k-1$ rows are defined by considering the columns from right to left, beginning at the vth column and continuing toward the first column. The $(v-k+2)$th row ends with $k-1$ ones, preceded by a zero, then a one, and the remaining columns have zeros; the $(v-k+3)$th row ends with $k-2$ ones, preceded by a zero, then two consecutive ones, and the remaining columns have zeros; and so on through the vth row which ends with a one, preceded by a zero, then $k-1$ ones, and the remaining columns have zeros.

Thus, for example, when $(v=8, k=3)$ and $(v=11, k=2)$, the preceding construction yields, respectively, the following matrices:

$$A = \begin{bmatrix} 1&1&1&0&0&0&0&0 \\ 0&1&1&1&0&0&0&0 \\ 0&0&1&1&1&0&0&0 \\ 0&0&0&1&1&1&0&0 \\ 0&0&0&0&1&1&1&0 \\ 0&0&0&0&0&1&1&1 \\ 0&0&0&0&1&0&1&1 \\ 0&0&0&0&1&1&0&1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1&1&0&0&0&0&0&0&0&0&0 \\ 0&1&1&0&0&0&0&0&0&0&0 \\ 0&0&1&1&0&0&0&0&0&0&0 \\ 0&0&0&1&1&0&0&0&0&0&0 \\ 0&0&0&0&1&1&0&0&0&0&0 \\ 0&0&0&0&0&1&1&0&0&0&0 \\ 0&0&0&0&0&0&1&1&0&0&0 \\ 0&0&0&0&0&0&0&1&1&0&0 \\ 0&0&0&0&0&0&0&0&1&1&0 \\ 0&0&0&0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&0&0&1&0&1 \end{bmatrix}.$$

It is clear that this construction produces for each $v \geq 2$ and each $k := 1, \ldots, \lfloor v/2 \rfloor$, a binary $v \times v$ matrix A which has k ones in each row and for which $C = k$. There remains to show that A is nonsingular; to do so, it will be convenient to refer to a certain submatrix B of A. Specifically, B is the $(k+1) \times (k+1)$ submatrix in the lower-right corner of A. Thus, for each of the two examples displayed above, the submatrix B is, respectively,

$$B = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

In general, B is a binary $(k+1) \times (k+1)$ circulant matrix having k ones in each row and defined as follows: the first row has k ones in the first k columns, and each remaining row is obtained from the preceding row by a cyclic shift of one column to the right. Thus, the general form for B is

$$B = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 & 1 \end{bmatrix}.$$

The structure of the matrix A facilitates the computation of $\det A$, whose value will serve to prove that A is nonsingular. To compute $\det A$, one expands along the main diagonal, beginning with the entry in position $(1,1)$ and continuing down through the entry in position $(v-k-1, v-k-1)$. This reveals that $\det A = \det B$. Next, in the matrix B, one removes the first row and places it as the last row. Thus,

$$\det A = \det B = \pm \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{vmatrix},$$

a $(k+1) \times (k+1)$ determinant whose value can be obtained from the Lemma; one finds that $\det A = \pm k \neq 0$, which proves that A is nonsingular. $\square$

The following notation will be helpful in proving the next theorem. In a square matrix, each of the two diagonals determines two triangles within the matrix. The triangular binary $n \times n$ matrices UL_n, UR_n, LL_n, and LR_n are defined by

$$UL_n := \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}, \qquad UR_n := \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

$$LL_n := \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}, \qquad \text{and} \qquad LR_n := \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

Thus, the notation indicates which of the four triangles in the matrix is made up of all ones. For example, UL_n has the upper-left triangle made up of all ones, and LR_n has the lower-right triangle consisting of all ones.

Theorem 4. *For each $v \geq 2$, there exists a nonsingular binary $v \times v$ matrix A which has row sum $k := \lfloor v/2 \rfloor$ and for which $C = k$. Thus, in this case, the upper bound k obtained in Theorem 1 is best possible for C.*

Proof. For each $v \geq 2$, the binary $v \times v$ matrix A is defined as follows. For $v = 2$, 3, 4, and 5 respectively, A is given by

$$A := \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right], \qquad A := \left[\begin{array}{c|cc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right],$$

$$A := \left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ \hline 0 & 1 & 1 & 0 \end{array}\right], \qquad \text{and} \qquad A := \left[\begin{array}{cc|ccc} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

For $v \geq 6$, A is defined similarly, according to the parity of v. If v is even,

$$A := \left[\begin{array}{ccc|ccc} & UL_{v/2} & & 0 & \cdots & 0 \\ 0 & \cdots & 0 & & LL_{v/2} & \\ \hline 0 & & & & & 0 \\ \vdots & LR_{v/2-1} & & & UR_{v/2-1} & \vdots \\ 0 & & & & & 0 \end{array}\right],$$

and if v is odd,

$$A := \left[\begin{array}{ccc|ccc} & & & 0 & \cdots & 0 \\ & UL_{\lfloor v/2 \rfloor} & & & LL_{\lfloor v/2 \rfloor} & \vdots \\ 0 & \cdots & 0 & & & 0 \\ \hline 0 & \cdots & 0 & 0 & & \\ \vdots & LR_{\lfloor v/2 \rfloor - 1} & & \vdots & UR_{\lfloor v/2 \rfloor} & \\ 0 & & & 0 & & \end{array}\right].$$

Thus, for example, when $v = 10$ and $v = 11$, one obtains, respectively,

$$A = \left[\begin{array}{ccccc|ccccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \qquad \text{and} \qquad A = \left[\begin{array}{ccccc|cccccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

It is clear that this construction produces for each $v \geq 2$, a binary $v \times v$ matrix A which has row sum $k := \lfloor v/2 \rfloor$ and for which $C = k$. There remains to show that A is nonsingular. To do so, one computes $\det A$ as follows, according to the parity of v.

If v is even, then the last column of A has exactly one 1, located in row $v/2+1$; one expands by this entry. Next, in the resulting array, the first column is subtracted from each of the other columns in the two left blocks. Then one expands by the entry in the upper-left corner. Now apply the following three-step algorithm:

1. add the last row to the row that has all -1s in the upper-left block; and
2. expand by the entry in the lower-left corner.
3. If there remain no rows having all -1s in the upper-left block of the resulting array, then the algorithm is finished. Otherwise go to the first step in the algorithm.

This shows that

$$\det A = \pm \begin{vmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{vmatrix},$$

a $(v/2-1) \times (v/2-1)$ determinant whose value can be obtained from the Lemma; one finds that $\det A = \pm k \neq 0$, which proves that A is nonsingular in this case.

If v is odd, then the first column is subtracted from each of the other columns in the two left blocks. Then one expands by the entry in the upper-left corner. Now apply the same three-step algorithm given above for the case when v is even. Finally, in the resulting array, the last row is removed and placed as the first row. This shows that

$$\det A = \pm \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{vmatrix},$$

a $(\lfloor v/2 \rfloor + 1) \times (\lfloor v/2 \rfloor + 1)$ determinant whose value can be obtained from the Lemma; one finds that $\det A = \pm \lfloor v/2 \rfloor = \pm k \neq 0$, which proves that A is nonsingular in this case also. $\square$

Remark 1. The constructions given in Theorems 3 and 4 are different in the following sense: if M and N are matrices of the same size and resulting respectively from the constructions given in Theorems 3 and 4, then one of M and N need not be obtainable from the other by means of row or column permutations. For example, let

$$M := \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad N := \left[\begin{array}{cc|ccc} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

Then it is not possible to change one of M and N to the other by permuting rows or columns. This is so because to effect such change, it is necessary that N have one row or column with exactly one 1. $\blacksquare$

Remark 2. From the proofs of Theorems 3 and 4, it might seem that if $C = k$ for a binary matrix A of constant row sum k, then $\{0, \pm k\}$ is the set of possible values for $\det A$. However, this is in general false. For example, each of maximal C value, the matrices

$$A_1 := \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad A_2 := \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

have, respectively, $(v = 6, k = 2, C = 2)$ and $(v = 8, k = 2, C = 2)$, but $\det A_1 = \det A_2 = 4$.

From the proofs of Theorems 3 and 4, it is also tempting to conjecture that for a binary matrix A of constant row sum k, the condition $\det A = k$ is sufficient to insure that $C = k$. But this is also false. For example, the matrix

$$A := \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

has $\det A = k = 3$, but $C = 2$, which is not maximal.

In fact, the relationship between the values of $\det A$ and C is unclear. For instance, of minimal C value, the matrix

$$A := \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

has $(v = 7, k = 3, C = 1)$ and $\det A = 24$. ■

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